Patterns of Euler's Totient Function

Cathal Stephens and Roonak Thapa

MIT PRIMES Circle

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Background Information

Definition

Integers are *coprime* to each other if their greatest common divisor is 1.

Example

- 9 and 4 are coprime because their greatest common divisor is 1.
- 15 and 10 are not coprime because their greatest common divisor is 5.

Definition

A *multiplicative function* $f(x)$ is a function on natural numbers such that $f(mn) = f(m)f(n)$, where $(m, n) = 1$.

Background Information

Definition

The *totient function* or φ-function is the number of positive integers less than or equal to some positive integer x that are coprime to x, denoted by $\varphi(n)$.

Example

If we list all positive integers less than or equal to 12, for $\varphi(12)$, we have the set $\{1, 2, 3...11, 12\}$. We can find $\varphi(12)$ by removing all non-coprime integers from our set, leaving $\{1, 5, 7, 11\}$, so $\varphi(12) = 4$.

Note: The totient function is multiplicative

Background Information

Definition

A function $f : X \to Y$ is *surjective* if for every element y in Y, there exists an element x in X such that $f(x) = y$.

Example

Consider the functions $f(x) = x^2$ and $g(x) = x^3$ both over the domain $[-1, 1]$ and the codomain $[-1, 1]$. We see that $f(x)$ is a not a surjective function as $f(x) = x^2$ is never negative. We see that $g(x)$ is a surjective function because for every y in the codomain $[-1, 1]$, there exists x in the domain $[-1, 1]$ such that $g(x) = y$.

Properties of the Totient Function

Proposition

For a prime number p and integer $s \geq 1$, $\varphi(p^s) = p^s - p^{s-1}$.

- Since p is prime, $\varphi(p^s) = \#\{1, 2, 3, \ldots, p^s\} - \#\{p, 2p, \ldots, p^s\}$
- The number of multiples of p, $\{p, 2p, \ldots, p^s\}$ can be written as $\frac{p^s}{p} = p^{s-1}$

Properties of the Totient Function

Proposition

For a positive integer x with prime factorization $x = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$
$\varphi(x) = (p_1^{e_1-1}(p_1 - 1))(p_2^{e_2-1}(p_2 - 1)) \cdots (p_r^{e_r-1}(p_r - 1))$.

- Since φ is multiplicative we can write, $\varphi(x) = \varphi(p_1^{e_1})\varphi(p_2^{e_2}) \cdots \varphi(p_r^{e_r})$
- Recall $\varphi(p_i^{e_i}) = p_i^{e_i} - p_i^{e_i-1}$
- Thus $\varphi(x) = (p_1^{e_1-1}(p_1 - 1))(p_2^{e_2-1}(p_2 - 1)) \cdots (p_r^{e_r-1}(p_r - 1))$

Codomain of the Totient Function

Proposition

For integers $x \geq 3$, $\varphi(x)$ is never odd. $\varphi(x)$ is only odd when $x = 2$.

- When x is a odd integer, $\varphi(x) = x - 1$ which is even.
- When x is a composite number, we can break down x into $p_1^{e_1} \cdot p_2^{e_2} \cdot ... \cdot p_k^{e_k}$, because of unique prime factorization so $\varphi(x) = \varphi(p_1^{e_1}) \cdot \varphi(p_2^{e_2}) \cdot \varphi(p_2^{e_2})$ which we know to all be evens or 1's. The product of evens and 1's is even so $\varphi(x)$ is a even number.

The codomain of the totient function does not include every even value

Proposition

The codomain of the totient function does not include every even value.

We computationally check every value $\varphi(x)$ for x in the interval $[15, 450]$, we found no value of x such that $\varphi(x) = 14$.

Lemma

There exists a lower bound for $\varphi(x)$ *where no value can* $\varphi(x) \geq \sqrt{\frac{x}{2}}$

We first determined that 14 would be the best value to search for. We found 15 to be the lower range because the $\varphi(x)$ is less than x, so the lowest value x that could return 14 is 15. The higher value is 450 because we set $\sqrt{\frac{x}{2}} = 15$ and found that $x = 450$.

The codomain of the totient function does not include every even value

In the program we have 2 lists, a list of all values less than x (valueList), and all lists that are divisors to x (div). When the program determines what x is, the program fills valueList with all the values of less than x, finds and stores all divisor in div, then checks if a value in valueList is in div. If the values are not coprime, that value in valueList is removed. Afterwards the remaining values are counted up and printed out, and the process restarts.

Based on the previous two propositions, the codomain of the totient function consists of only but not all even numbers and 1.

When does $\varphi(x_1) = \varphi(x_2) \cdots = \varphi(x_n)$?

- Consider $\varphi(15) = \varphi(30)$
- $\varphi(15) = (3-1)(5-1) = 8$, $\varphi(30) = (2-1)(3-1)(5-1) = 8$
- Given some positive integer n, can we find n distinct positive integers $x_1, x_2, \ldots, x_n$ such that $\varphi(x_1) = \varphi(x_2) = \cdots = \varphi(x_n)$? What patterns can we identify?

When does $\varphi(x_1) = \varphi(x_2) \cdots = \varphi(x_n)$?

Recall $\varphi(p^r) = p^{r-1}(p-1)$

Example

Consider $\varphi(2 \cdot 3 \cdot 5) = (2-1)(3-1)(5-1) = 8$

- $(2-1)(3-1)(5-1) = (3-1)(5-1) = \varphi(3 \cdot 5)$
- $(2-1)(3-1)(5-1) = 2(2-1)(5-1) = \varphi(2^2 \cdot 5)$
- $(2-1)(3-1)(5-1) = 2^2(2-1)(3-1) = \varphi(2^3 \cdot 3)$

- For $\varphi(x) = \varphi(p_1 p_2 \cdots p_r)$, $n = 1 + r$

When does $\varphi(x_1) = \varphi(x_2) \cdots = \varphi(x_n)$?

Example

- $\varphi(2 \cdot 3 \cdot 5) = (2-1)(3-1)(5-1) = 2^3(2-1) = \varphi(16) = 8$

Lemma (Bertrand's postulate)

For two consecutive primes p_n, p_{n+1}, $p_{n+1} < 2p_n$.

- We can remove any two consecutive primes from the prime factorization of $\varphi(x)$
- For $\varphi(x) = \varphi(p_1 p_2 \cdots p_r)$, $n = 1 + r + r - 2$

When does $\varphi(x_1) = \varphi(x_2) \cdots = \varphi(x_n)$?

Theorem

Given a positive integer n, we can find a set $\{x_1, \ldots, x_n\}$ where $\varphi(x_i) = m$ for some integer m and $1 \leq i \leq n$.

Construction:

- $\varphi(p_1 p_2 \cdots p_r) = m$
- $\varphi(p_1^{e_1} p_2^{e_2} \cdots p_{i-1}^{e_{i-1}} \hat{p}_i \cdots p_r) = m$ for $1 \leq i \leq r$
- Where $e_j = 1 + f_{i,j}$, $1 \leq j \leq i-1$
- $\varphi(p_1^{e_1} p_2^{e_2} \cdots p_{i-1}^{e_{i-1}} \hat{p}_i \hat{p_{i+1}} \cdots p_r) = m$ for $2 \leq i \leq r-1$
- Where $e_j = 1 + f_{i,j} + f_{i+1,j}$, $1 \leq j \leq i-1$
- Thus $n = 1 + r + r - 2$